Estimation of the Condition Number of a Square Matrix with Hager's Method

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Abstract

The condition number of a matrix quantifies how accurately the solution to **Ax = b** is computed. More specifically, it equals the maximum by which a relative error in the right-hand side b can be magnified into the relative accuracy of the solution. In this study we examine Hager's method for estimating the condition number of a matrix when using the one-norm and infinity norm.

Background Information

Conditioning

Consider solving a linear system of equation

$$Ax = b$$

In reality, the input is not accurate, so what we actually solve maybe the equation below

$$A(x + \delta x) = b + \delta b$$

We want to explore how the relative error on b can effect the relative error on x

$$\frac{||\delta x||}{||x||} \le \kappa(A, b, \delta b)\frac{||\delta b||}{||b||}$$

Definitions

1 Vector norm: For vector $v = (v_0, v_1 \dots v_{n-1})^T$ define l_p norm as

$$||v||_p = (\sum_{i=0}^{n-1} (v_i)^p)^{1/p}$$

2 Matrix norm: For matrix A_{m*n} ,define p norm of a matrix as

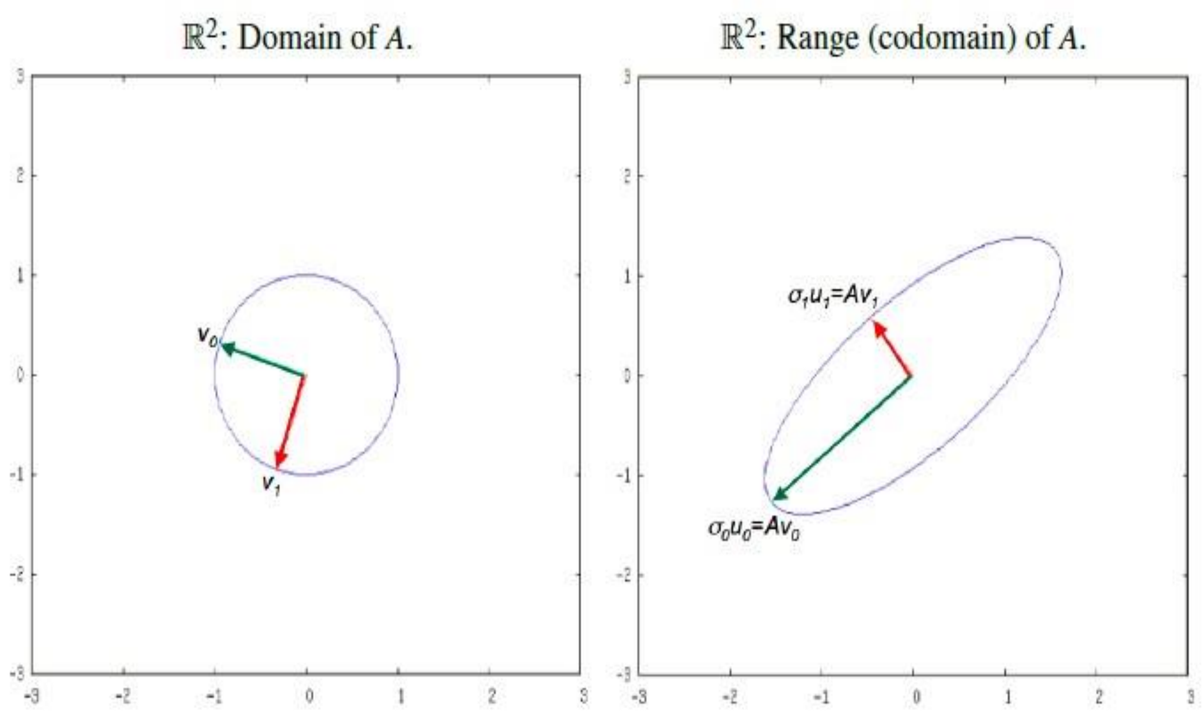
$$||A||_p = max\frac{||Ax||_p}{||x||_p}$$

Theorem:

For any matrix A, we have

$$||A||_1 = ||A^T||_\infty$$

Condition Number



$||A||$ Maximum Magnification

$\frac{1}{||A^{-1}||}$ Inverse of the Minimal Magnification

The ratio :the effect of small change at rhs has on the lhs

We define the **condition number** as

$$K(A) = ||A|| * ||A^{-1}||$$

Methodology and Algorithm

Motivations

(1) Computing Inverse matrix expensive $O(n^3)$

(2) LAPACK require Pass in 1-norm when computing Condition Number

```
      SUBROUTINE DGECON( NORM, N, A, LDA, ANORM, RCOND, WORK, IWORK,
     $                   INFO )
```

Condition Number function header in LAPACK

(3) GOTO Considered Harmful

```
     $                   GO TO 20
            CALL DRSCL( N, SCALE, WORK, 1 )
         END IF
         GO TO 10
      END IF
```

GOTO In LAPACK Routine

Algorithm for Computing L_1 norm

Input: Matrix A, dimension n

(0) Compute $||A||_1 = \max \sum_{i=1}^{n} ||A_{ij}||_1$

(1) Pick the vector $x = \frac{1}{n}(1,1,1 \dots 1)^T$

(2) Factor $A = LU$

(3) Solve $Lb = x$ and $Uy = b$

(4) Set $v = sign(A^{-1}x) = sign(y)$

(5) Solve $U'd = y$ and $L'g = d$

(6)If $||g_\infty|| < g^T x$, then estimate as

$$est(||A^{-1}||_1) = ||A^{-1}x||_1$$

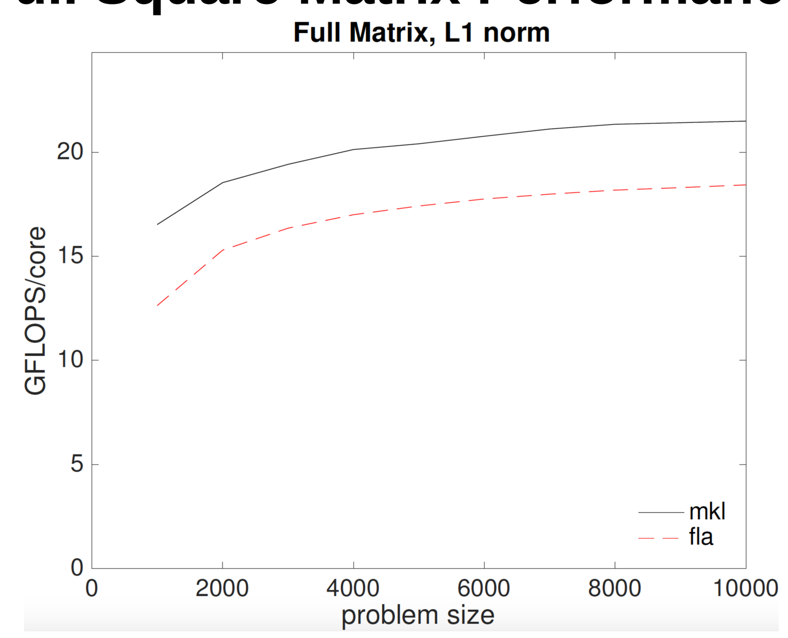
(7) else set $x = e_j$ where $g_j = g_\infty$ and repeat from (3)

(8) $K(A) = ||A|| * ||A^{-1}||$

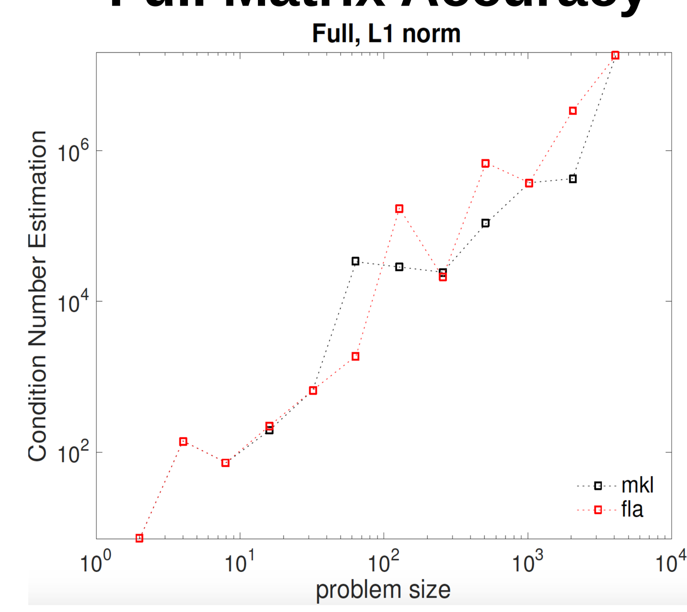
Performance and Accuracy

All data coming from TACC Stampede System (Intel® Xeon® CPU E5-2680, Sandy Bridge, 23.76 GFLOPS peak for single-core, 21.6 GFLOPS/core peak for multi-core) processor using Intel C compiler version 15.0.2 with optimization flag -O2.

Full Square Matrix Performance

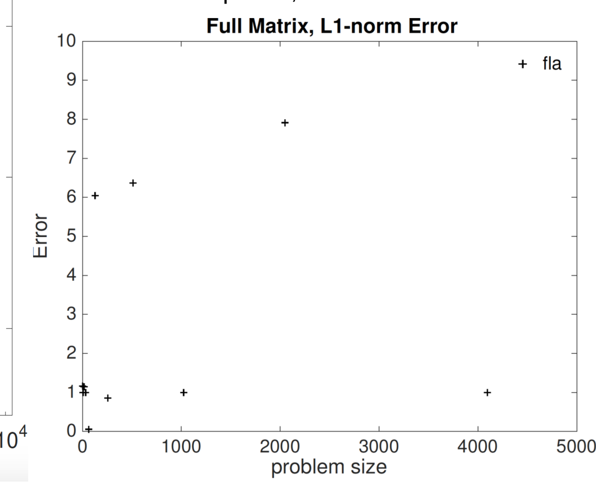


Full Matrix Accuracy

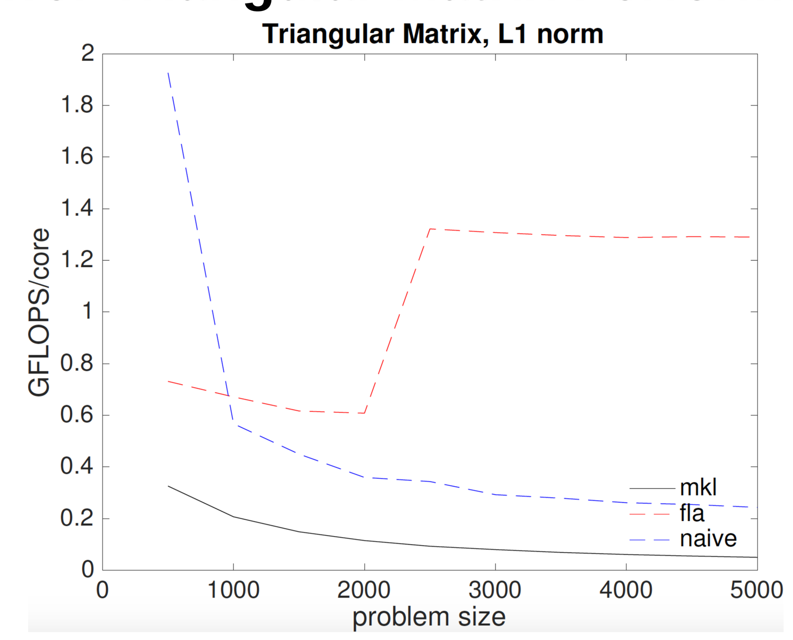


Error

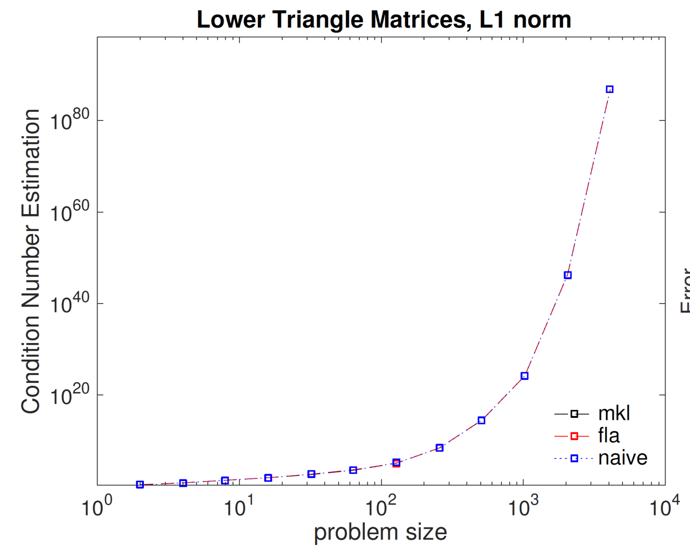
Y axis reflects esti cond/lapack cond
Want same order
0.9~10 is acceptable,1 reflects exact match



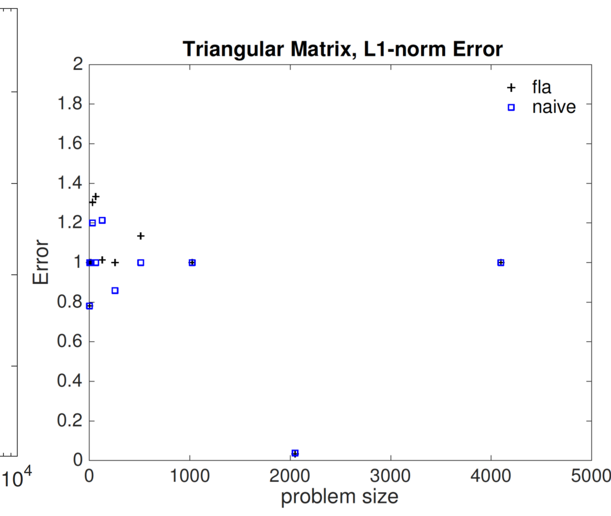
Lower Triangular Matrix Performance



Lower Triangular Matrix Accuracy



Error



Conclusion

Performance

Full Matrices
- MKL Better than FLA

Triangular Matrices
- Naïve C extremely good when small
- FLA good in general.
- MKL generally stable

Accuracy

Full Matrices
- FLA estimates good

Triangular Matrices
- Generally good with bad on one of them

Future Work

1 Extend to complex matrices
2 Extend to full matrices of C implementation
3 Find the condition number under l_2 norm
4 Deal with both overflow and underflow

Reference

https://github.com/flame/blis
https://github.com/flame/libflame
http://www.netlib.no/netlib/lapack/double/dgecon.f
Nicholas J. Higham A Survey of Condition Number Estimation for Triangular Matrices(1987)
Robert A. van de Geijn, Linear Algebra: Foundations to Frontiers A Collection of Notes on Numerical Linear Algebra

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